Generalized closed sets: a survey of recent work

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ABSTRACT. We present an overview of our recent research in the field of generalized closed sets of a topological space.

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1 Introduction

The notion of closed set is fundamental in the study of topological spaces. In 1970, Levine [15] introduced the concept of generalized closed sets in a topological space by comparing the closure of a subset with its open supersets. He defined a subset A of a topological space X to be *generalized closed* (briefly, *g-closed*) if $\mathrm{cl}A \subseteq U$ whenever $A \subseteq U$ and U is open. Note that this definition uses both the "closure operator" and "openness" of the superset. By considering other generalized closure operators or classes of generalized open sets, various notions analogous to Levine's g-closed sets have been considered, refer to [5] for more detail.

The study of generalized closed sets has produced some new separation axioms which lie between T_0 and T_1, such as $T_{\frac{1}{2}}$, T_{gs} and $T_{\frac{3}{4}}$. Some of these properties have been found to be useful in computer science and digital topology [14]. Recent work by Cao, Ganster and Reilly has shown that generalized closed sets can also be used to characterize certain classes of topological spaces and their variations, for example the class of extremally disconnected spaces and the class of submaximal spaces, see [3] and [4]. For convenience, we provide definitions of eleven classes of generalized closed sets in Definitions 1.1 and 1.4 below.

Definition 1.1. Let X be a topological space. A subset A of X is called:
(i) α-closed if $\mathrm{cl}(\mathrm{int}(\mathrm{cl}A)) \subseteq A$;
(ii) semi-closed if $\mathrm{int}(\mathrm{cl}A) \subseteq A$;
(iii) preclosed if $\mathrm{cl}(\mathrm{int}A) \subseteq A$;
(iv) β-closed if $\mathrm{int}(\mathrm{cl}(\mathrm{int}A)) \subseteq A$.

[0]This paper is an expanded version of a lecture given at the Conference on General and Geometric Topology, Research Institute for Mathematical Sciences, Kyoto University, October 2001. It represents joint work with J. Cao, M. Ganster, S. Greenwood and Ch. Konstadilaki (see References).

Definition 1.2. Let X be a topological space. A subset A of X is called:
(i) α-open if $X \setminus A$ is α-closed, or equivalently, if $A \subseteq \mathrm{int}(\mathrm{cl}(\mathrm{int}A))$;
(ii) semi-open if $X \setminus A$ is semi-closed, or equivalently, if $A \subseteq \mathrm{cl}(\mathrm{int}A)$;
(iii) preopen if $X \setminus A$ is preclosed, or equivalently, if $A \subseteq \mathrm{int}(\mathrm{cl}A)$;
(iv) β-open if $X \setminus A$ is β-closed, or equivalently, if $A \subseteq \mathrm{cl}(\mathrm{int}(\mathrm{cl}A))$.

We note that the collection of all α-open subsets of X is a topology on X, called the *α-topology* [19], which is finer than the original one. We denote X with its α-topology by X_α. A set $A \subseteq X$ is α-open if and only if A is semi-open and preopen [20]. Some authors use the term *semi-preopen* (*semi-preclosed*) for β-open (β-closed).

Definition 1.3. Let X be a topological space, and suppose $A \subseteq X$:
(i) the *α-closure of* A, denoted by $\mathrm{cl}_\alpha A$, is the smallest α-closed set containing A;
(ii) the *semi-closure of* A, denoted by $\mathrm{cl}_s A$, is the smallest semi-closed set containing A;
(iii) the *preclosure of* A, denoted by $\mathrm{cl}_p A$, is the smallest preclosed set containing A;
(iv) the *β-closure of* A, denoted by $\mathrm{cl}_\beta A$, is the smallest β-closed set containing A.

It is well-known that $\mathrm{cl}_\alpha A = A \cup \mathrm{cl}(\mathrm{int}(\mathrm{cl}A))$, $\mathrm{cl}_s A = A \cup \mathrm{int}(\mathrm{cl}A)$, $\mathrm{cl}_p A = A \cup \mathrm{cl}(\mathrm{int}A)$ and $\mathrm{cl}_\beta A = A \cup \mathrm{int}(\mathrm{cl}(\mathrm{int}A))$.

Definition 1.4. Let X be a topological space. A subset A of X is called:
(i) generalized closed (briefly, *g-closed*) [15] if $\mathrm{cl}A \subseteq U$ whenever $A \subseteq U$ and U is open;
(ii) semi-generalized closed (briefly, *sg-closed*) [2] if $\mathrm{cl}_s A \subseteq U$ whenever $A \subseteq U$ and U is semi-open;
(iii) generalized semi-closed (briefly, *gs-closed*) [1] if $\mathrm{cl}_s A \subseteq U$ whenever $A \subseteq U$ and U is open;
(iv) generalized α-closed (briefly, *gα-closed*) [16] if $\mathrm{cl}_\alpha A \subseteq U$ whenever $A \subseteq U$ and U is α-open, or equivalently, if A is g-closed with respect to the α-topology;
(v) α-generalized closed (briefly, *αg-closed*) [17] if $\mathrm{cl}_\alpha A \subseteq U$ whenever $A \subseteq U$ and U is open;
(vi) gp-closed [18] if $\mathrm{cl}_p A \subseteq U$ whenever $A \subseteq U$ and U is open;
(vii) gsp-closed [8] if $\mathrm{cl}_\beta A \subseteq U$ whenever $A \subseteq U$ and U is open.

A subset A of X is *g-open* [15] (*sg-open* [2]) if $X \setminus A$ is g-closed (sg-closed). Other classes of generalized open sets are defined in a similar manner.

Recall that a space (X, τ) is called *resolvable* if there exists a pair of disjoint dense subsets in X. Otherwise X is called *irresolvable.* (X, τ) is said to be *strongly irresolvable* if every open subspace is irresolvable. Hewitt [12] has shown that every space (X, τ) has a decomposition $X = F \cup G$, where F is closed and resolvable and G is open and hereditarily irresolvable. We shall call this decomposition the *Hewitt decomposition* of (X, τ). There is another important decomposition of a space which we shall call the *Jankovic–Reilly decomposition.* Since every singleton $\{x\}$ of a space (X, τ) is either nowhere dense or preopen (see [13]), we clearly have $X = X_1 \cup X_2$, where $X_1 = \{x \in X : \{x\}$ is nowhere dense $\}$ and $X_2 = \{x \in X : \{x\}$ is preopen $\}$.

Remark 1.5. *Throughout this paper, F and G will always refer to the Hewitt decomposition, and X_1 and X_2 always refer to the Jankovic-Reilly decomposition.*

In [9], Dontchev summarized the relationships between most of the notions of Definitions 1.1 and 1.4 in a diagram, Figure 1. He also pointed out that none of the implications can be reversed in general.

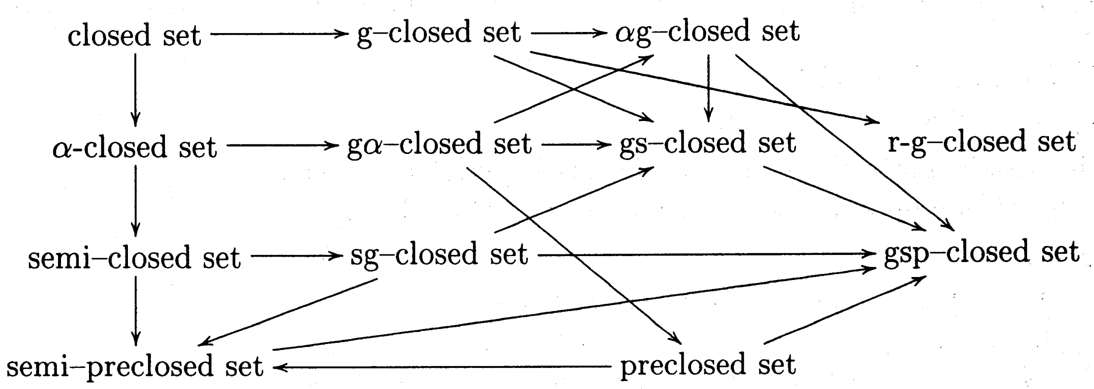

Figure 1

We will require the following classes of topological spaces.

Definition 1.6. Let X be a topological space. X is:
(i) T_{gs} [17] if every gs-closed subset of X is sg-closed; or equivalently, if for each $x \in X$, $\{x\}$ is either closed or preopen [3];
(ii) $T_{\frac{1}{2}}$ [15] if every g-closed subset of X is closed; or equivalently, if for each $x \in X$, $\{x\}$ is either closed or open [11];
(iii) semi-$T_{\frac{1}{2}}$ [2] if every singleton is either semi-open or semi-closed in X;
(iv) nodec [10] if every nowhere dense set of X is closed;
(v) nodeg if every nowhere dense set of X is g-closed;
(vi) extremally disconnected [3] if the closure of each open set of X is open;
(vii) g-submaximal [4] if every dense subset of X is g-open;
(viii) sg-submaximal [3] if every dense subset of X is sg-open.

2 Results

Our starting point in the investigation of generalized closed sets was two open questions that Dontchev posed in [9], namely :

Characterize those spaces where

(A) Every semi–preclosed set is sg–closed, and

(B) Every preclosed set is gα–closed.

These questions have been solved by Cao, Ganster and Reilly in [4]. To our surprise, both decompositions mentioned before, i.e. the Hewitt decomposition and the Jankovic–Reilly decomposition, played a key role in our solution to these questions. Further studies have shown that these decompositons are important in many more questions concerning generalized closed sets.

Recall that a space (X, τ) is said to be *locally indiscrete* if every open subset is closed.

Theorem 2.1. *[4]*

For a space (X,τ) the following are equivalent:
(1) (X,τ) satisfies (A),
(2) $X_1 \cap sclA \subseteq spclA$ for each $A \subseteq X$,
(3) $X_1 \subseteq int(clG)$,
(4) (X,τ) is the topological sum of a locally indiscrete space and a strongly irresolvable space,
(5) (X,τ) satisfies (B),
(6) (X,τ^α) is g-submaximal.

This result motivated us to look for other possible converses in Figure 1. Out of the many results we obtained we shall present just three here.

Theorem 2.2. *[3]*
For a space (X,τ) the following are equivalent:
(1) every semi–preclosed set is gα–closed,
(2) (X,τ^α) is extremally disconnected and g–submaximal.

Theorem 2.3. *[3]*
For a space (X,τ) the following are equivalent:
(1) $X_1 \subseteq clG$,
(2) every preclosed subset is sg–closed,
(3) (X,τ) is sg–submaximal,
(4) (X,τ^α) is sg–submaximal.

Corollary 2.4. If (X,τ^α) is g–submaximal then (X,τ^α) is also sg–submaximal. The converse, however, is false (see [3]).

3 Lower Separation Axioms

The closer investigation of generalized closed sets has had significant impact on the theory of separation axioms. In Figure 1, the search for converses of other implications leads to the consideration of certain lower separation axioms.

For example Maki et al. [17] have called a space (X,τ) a T_{gs} space if every gs–closed subset is sg–closed. We have been able to characterize T_{gs} spaces in the following way.

Theorem 3.1. *[5]*
For a space (X,τ), the following are equivalent:
(1) (X,τ) is a T_{gs} space,
(2) every nowhere dense subset of (X,τ) is a union of closed subsets,
(3) every gsp–closed set is semi–preclosed, i.e. (X,τ) is semi–pre–$T_{1/2}$ [8],
(4) every singleton of (X,τ) is either preopen or closed.

Theorem 3.2. *[5]*
For a space (X,τ) the following are equivalent:
(1) Every gα–closed set is g–closed,
(2) every nowhere dense subset is locally indiscrete as a subspace,
(3) every nowhere dense subset is g–closed,
(4) every α–closed set is g–closed.

Observe, however, that there are spaces in which every nowhere dense subset is g–closed but there exists a nowhere dense set which is not closed (see [5]).

4 Gp–closed Sets

Definition 4.1. *A subset A of a space (X, τ) is called* generalized preclosed, *briefly gp*–closed, *[18] if $pcl A \subseteq U$ whenever $A \subseteq U$ and U is open.*

Our study of generalized preclosed sets has been carried out in [6]. As one might expect, here also the Hewitt decomposition, the Jankovic–Reilly decomposition, submaximality and extremal disconnectedness play a significant role. Out of the many results that we obtained we mention here two important characterizations.

Theorem 4.2. *[6]*
For a space (X, τ) the following are equivalent :
(1) (X, τ) is a T_{gs}-space,
(2) Every gp–closed subset of (X, τ) is preclosed,
(3) Every gsp–closed subset of (X, τ) is semi–preclosed,
(4) Every gp–closed subset of (X, τ) is semi–preclosed.

Theorem 4.3. *[6]*
For a space (X, τ) the following are equivalent :
(1) Every gsp–closed subset of (X, τ) is gp–closed,
(2) Every semi–preclosed subset of (X, τ) is gp–closed,
(3) (X, τ) is extremally disconnected.

5 A unified approach: qr-closed sets

An enlarged and enhanced version of Figure 1 was provided by Cao, Greenwood and Reilly [7]. We label it Figure 2. The relationships between these classes of subsets is much clearer in Figure 2 than in Figure 1.

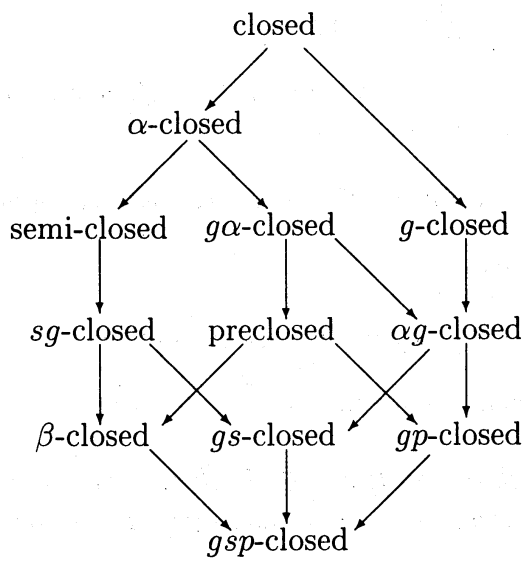

Figure 2

We address two general questions. Each generalization in Definition 1.4 involves a closure operation and a notion of "openness". Specifically, each definition involves either cl, cl_α, cl_s, cl_p, or cl_β of A together with U being either open, α-open, or semi-open. The first question, which arises from these definitions, is: do any new classes of generalized closed sets exist if we consider every possible pairing of the five closure operations mentioned above with the notions of openness in Definition 1.2? In order to study each possible pairing in a unified way, Cao, Greenwood and Reilly [7] introduced the term *qr-closed*, where q represents a closure operation, and r represents a notion of generalized openness. Surprisingly, in most cases, they obtained new characterizations of existing classes. These cases provide new insights into the nature of generalized closed sets.

As noted above, Figure 2 summarises the known relationships between classes of generalized closed sets. In general, none of the implications represented in the diagram is reversible. The second question we will consider is: are the implications represented in the diagram the only implications which apply in general? As a consequence of answering these two questions, we will derive new relationships between different types of qr-closed sets which characterize certain topological spaces.

In the following we shall denote closed (resp. semi-closed, preclosed) by τ-closed (resp. s-closed, p-closed), and cl$A$ by $\mathrm{cl}_\tau A$ for $A \subseteq X$, whenever it is convenient to do so. Similarly we denote open (resp. semi-open, preopen) by τ-open (resp. s-open, p-open). Let $\mathcal{P} = \{\tau, \alpha, s, p, \beta\}$.

Definition 5.1. Let X be a topological space and $q, r \in \mathcal{P}$. A subset $A \subseteq X$ is called *qr-closed* if $\mathrm{cl}_q A \subseteq U$ whenever $A \subseteq U$ and U is r-open.

We note that each type of generalized closed set in Definition 1.4 is defined to be qr-closed for some $q, r \in \mathcal{P}$. A set A is g-closed if it is $\tau\tau$-closed, αg-closed if it is $\alpha\tau$-closed, gs-closed if it is $s\tau$-closed, gp-closed if it is $p\tau$-closed, gsp-closed if it is $\beta\tau$-closed, $g\alpha$-closed if it is $\alpha\alpha$-closed, and sg-closed if it is ss-closed.
The proof of the following lemma is straightforward.

Lemma 5.2. *If X is a topological space, $A \subseteq X$, and $q \in \mathcal{P}$, then $x \in \mathrm{cl}_q A$ if and only if for each q-open set G, with $x \in G$, $G \cap A \neq \emptyset$.*

The following lemma gives two useful decompositions of a topological space.

Lemma 5.3. *Let X be a topological space.*

(i) [13] Every singleton of X is either preopen or nowhere dense.

(ii) Every singleton of X is either open or preclosed.

Theorem 5.4. *Let X be a topological space. If $q, r \in \mathcal{P}$, then every qr-closed subset of X is q-closed if and only if each singleton of X is either q-open or r-closed.*

Corollary 5.5. *Let X be a topological space, and let $A \subseteq X$ be a subset. If $r \in \{p, \beta\}$ then A is:*

(i) τr-closed if and only if it is closed;

(ii) αr-closed if and only if it is α-closed;

(iii) sr-closed if and only if it is semi-closed.

Furthermore, if $r \in \{\alpha, s, p, \beta\}$ then A is:

(iv) pr-closed if and only if it is preclosed;

(v) βr-closed if and only if it is β-closed.

Theorem 5.6. *Let X be a topological space. Then a subset A of X is $q\alpha$-closed if and only if A is qs-closed, for any $q \in \mathcal{P}$.*

Cao, Greenwood and Reilly [7] showed that for each $q, r \in \mathcal{P}$, the property qr-closed is equivalent to a known type of generalized closed set, except when $q = \tau$ and $r = \alpha$ (or equivalently $r = s$). They established that the class of $\tau\alpha$-closed sets is in fact new. By definition, each closed set is $\tau\alpha$-closed and each $\tau\alpha$-closed set is both g-closed and $g\alpha$-closed.

Theorem 5.7. *Let X be a topological space. Then the following statements are equivalent:*

(i) every $\tau\alpha$-closed subset of X is closed;

(ii) every $\tau\alpha$-closed subset of X is α-closed;

(iii) every $\tau\alpha$-closed subset of X is semi-closed;

(iv) every $g\alpha$-closed subset of X is semi-closed;

(v) X is a semi-$T_{\frac{1}{2}}$ space.

Theorem 5.8. *Let X be a topological space. Then every g-closed subset of X is $\tau\alpha$-closed if and only if X is a T_{gs}-space.*

Theorem 5.9. *Let X be a topological space. Then the following statements are equivalent:*

(i) X is nodec;

(ii) each α-closed subset of X is closed;

(iii) each $g\alpha$-closed subset of X is $\tau\alpha$-closed;

(iv) each α-closed subset of X is $\tau\alpha$-closed.

From Theorems 5.7, 5.8 and 5.9, we see that in a general topological space $\tau\alpha$-closed sets are not equivalent to closed sets, g-closed sets, $g\alpha$-closed sets, semi-closed or α-closed

6 Relationships

We now consider the completeness of Figure 2. We will introduce a new relationship not present in Figure 2, and establish that no other relationships exist in the general case. It follows from Theorem 5.6 that every $g\alpha$-closed set is sg-closed. This implication cannot be reversed in general by the following theorem.

Theorem 6.1. *Let X be a topological space. Each sg-closed subset of X is $g\alpha$-closed if and only if X is extremally disconnected.*

Next we establish that no further relationships exist in general. First we confirm that in general none of the implications in Figure 2 can be reversed. With the exception of two cases, it has been shown that the reverse implications occur only if the space has a specific property [3], [4], [5], [6], [9]. Theorem 6.2 below addresses one of the remaining cases. The other generates a new topological property defined in 6.3 below.

Theorem 6.2. *Let X be a topological space. Then X is nodeg if and only if every αg-closed subset of X is g-closed.*

Cao, Greenwood and Reilly [7] defined a new class of topological spaces.

Definition 6.3. A space X is defined to be βgs if every gsp-closed subset of X is gs-closed.

It is shown in [6] that X is a βgs-space if and only if every β-closed subset of X is gs-closed. The following implications follow from definitions and characterizations of g-submaximality of X_α in [4].

$$X_\alpha \text{ is } g\text{-submaximal} \to X \text{ is } \beta gs \to X \text{ is } sg\text{-submaximal}$$

Note that if X is a T_{gs}-space, the lefthand arrow is reversible; and if X is extremally disconnected, then the righthand arrow is reversible. Neither of these two arrows is reversible in general. In fact, the space X defined in Example 3.5 of [3] is sg-submaximal, but not βgs. The following two examples of finite spaces distinguish between these three classes of spaces.

Example 6.4. Let $X = \{a, b, c, d\}$, and let $\tau = \{\phi, X, \{a\}, \{c, d\}, \{a, c, d\}\}$. Then X is sg-submaximal, but it is not a βgs-space, since $\{a, c\}$ is β-closed but not gs-closed.

Example 6.5. Let $X = \{a, b, c, d, e\}$ and let

$$\mathcal{B} = \{\{b\}, \{d\}, \{a, b\}, \{d, e\}, \{b, c, d, e\}\}$$

be a base for a topology on X. Then X is a βgs-space, but X_α is not g-submaximal since $\{a, b, c, d\}$ is dense in X_α, but not g-open in X_α.

No further relationships exist in general by [3], [4], [5], [6], Theorems 5.7, 5.8, 5.9, 6.1, 6.2, and the following two theorems from [7].

Theorem 6.6. *Let X be a topological space. Then the following statements are equiva-*

(i) X *is a* T_{gs}*-space;*

(ii) *every* g*-closed subset of* X *is* $g\alpha$*-closed;*

(iii) *every* αg*-closed subset of* X *is* sg*-closed;*

(iv) *every* g*-closed subset of* X *is* β*-closed;*

(v) *every* g*-closed subset of* X *is* p*-closed;*

(vi) *every* g*-closed subset of* X *is* sg*-closed;*

(vii) *every* αg*-closed subset of* X *is* β*-closed;*

(viii) *every* gs*-closed subset of* X *is* β*-closed.*

Theorem 6.7. *Let* X *be a topological space. Then:*

(i) X *is extremally disconnected if and only if every semi-closed subset of* X *is* αg*-closed;*

(ii) X *is extremally disconnected if and only if every* sg*-closed subset of* X *is* αg*-closed;*

(iii) X *is nodeg and extremally disconnected if and only if every semi-closed subset of* X *is* g*-closed;*

(iv) X *is* $T_{\frac{1}{2}}$ *if and only if every* $g\alpha$*-closed subset of* X *is semi-closed;*

Thus we have a new diagram, Figure 3 below, showing all relationships between the classes of generalized closed sets under discussion. None of the implications shown in Figure 3 can be reversed in general topological spaces.

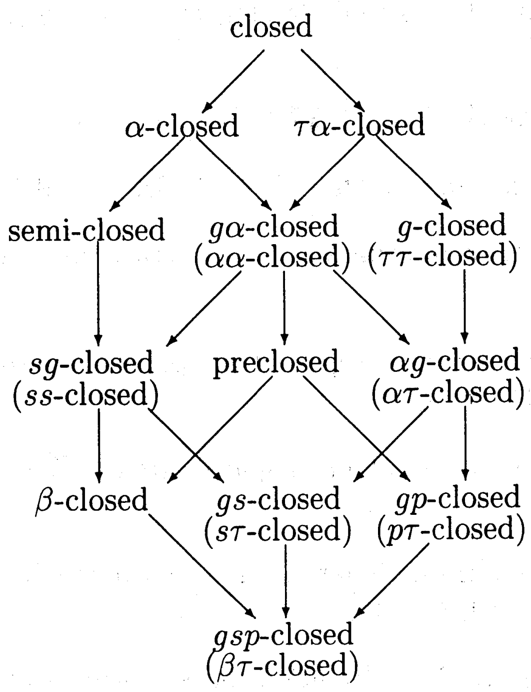

Figure 3

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